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March 2010

Compiled: February 10, 2014

Advanced econometric theory
Exercises 13
Equality constraints

1. We consider a dynamic model of the form

$$y_t = \theta_0 y_{t-1} + \theta_1 x_t + \theta_2 x_{t-1} + u_t \,,\ t = 1, \dots, T.$$

   We wish to test the hypothesis that this model can can be written as a non-dynamic regression model with $AR(1)$ errors, *i.e.*

$$y_t - \theta_0 y_{t-1} = a\,(x_t - \theta_0 x_{t-1}) + u_t,\ t = 1, \dots, T.$$

   Write the constraints entailed by the latter model:

   (a) in explicit form;
   (b) in implicit form;
   (c) in mixed form.

   Using the definitions of these three types of formulation, explain your answers.

2. Consider the linear model:

$$y_i = x_i'\theta + u_i, i = 1, \dots, n$$

   where x_i is a $p \times 1$ fixed vector such that the matrix $x = [x_1', x_2', \dots, x_n']'$ has rank p, and the u_i are random disturbances such that

$$\begin{aligned} E(u_i) &= 0, & i &= 1, \dots, n \\ E(u_i\, u_i) &= \sigma^2, & \text{if } i &= j \\ &= 0, & \text{if } i &\neq j. \end{aligned}$$

Further, we consider the following explicit linear constraints : $\exists a \in \mathbb{R}^q$ such that

$$\theta = Ha + h,$$

where H is a $p \times q$ matrix with rank q, $1 \leq q < p$, and h is a $q \times 1$ vector.

(a) Express the above constraint in implicit form.

(b) Show that the ordinary least squares (OLS) estimators of θ, based on explicit and implicit constraints, are identical.

(c) Show that the constrained OLS estimator of θ is more precise (in the sense that that its covariance matrix is smaller) than the unconstrained OLS estimator of θ.

(d) Let $\hat{\theta}^0$ and $\hat{\theta}$ be the constrained and unconstrained estimators of θ. Show that $\hat{\theta}^0$ and $\hat{\theta} - \hat{\theta}^0$ are uncorrelated.

3. Let $L_n(\theta)$ be a likelihood function such that the maximum likelihood (ML) $\hat{\theta}_n$ strongly converges to θ_0, and

$$\frac{1}{\sqrt{n}} \frac{\partial L_n}{\partial \theta}(\theta_0) \underset{n \to \infty}{\overset{d}{\to}} N[0, I_0] \ ,$$

$$-\frac{1}{n} \frac{\partial^2 L_n}{\partial \theta \, \partial \theta}(\theta_0) \underset{n \to \infty}{\overset{p.s.}{\to}} J_0 \, ,$$

where I_0 and J_0 are positive definitive matrices. Consider the mixed constraint:

$$g(\theta, a) = 0 \text{ for some } a \in \mathbb{R}^q$$

where $g(\theta, a)$ is an $r \times 1$ vector, $\partial g / \partial \theta'$ has r, and $\partial g / \partial a'$ has rank q. Let $\hat{\theta}_n^0$ be the constrained ML estimator of θ.

(a) Show that the random vector $\sqrt{n}\left(\hat{\theta}_n^0 - \theta_0\right)$ is asymptotically equivalent to a linear transformation of $\sqrt{n}\left(\hat{\theta}_n - \theta_0\right)$.

(b) Determine the asymptotic covariance matrix of $\sqrt{n}\left(\hat{\theta}_n^0 - \theta_0\right)$ when

(1) $g(\theta, a) = \theta - h(a)$,

(2) $g(\theta, a) = g(\theta)$.

(c) Show that $\sqrt{n}\left(\hat{\theta}_n^0 - \theta_0\right)$ and $\sqrt{n}\left(\hat{\theta}_n - \hat{\theta}_n^0\right)$ are asymptotically uncorrelated when $I_0 = J_0$.

4. Under the same conditions as in question 3, show that the estimator $\hat{\theta}_n^0$ obtained by minimizing $\left(\hat{\theta}_n - \theta\right)' \tilde{J}_n \left(\hat{\theta}_n - \theta\right)$ with respect to θ and a under the constraint $g\left(\theta, a\right) = 0$ is asymptotically equivalent to $\hat{\theta}_n^0$, when $\tilde{J}_n$ converges (with probability one) to J_0.